RECURRENCE-TRANSIENCE BOUNDARY FOR 1-D RANDOM WALK

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Abstract. The recurrence condition for irreducible 1-D nearest neighbor random walk is characterized by

$$\lim_{n\to\infty} \frac{q_n}{p_n} = \ell$$

where q_n and p_n are the transition probabilities to the left and right respectively at site n. The random walk is recurrent if $\ell > 1$ and transient if $\ell < 1$ (see [5, corollary 5.2.1], [1, Theorem 1]). We investigate the case where $\ell \to 1$ from above and below to understand the asymptotic boundary of recurrence for 1-D random walk on $\mathbb{Z}$.

1. Introduction

It is known that the recurrence condition for irreducible 1-D nearest neighbor random walk is characterized by the following quantity:

$$S := \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{q_k}{p_k} \tag{1.1}$$

where q_n and $p_n := 1 - q_n$ are the jump probabilities to the left and right respectively. See for example [1, theorem 1] or [5, proposition 5.2.2]. We remark that there is a 'lazy' variant to the random walk, where it has probability $r_n > 0$ to not jump. The recurrence-transience condition for such a 'lazy' walk is the same (see [5, section 5.2]) for walks on both $\mathbb{Z}$ and $\mathbb{Z}_+$. So to simplify matters, we restrict ourselves to ***irreducible 'non-lazy' birth-death chain***.

A random walk is said to be ***recurrent*** if $S = \infty$ and ***transient*** otherwise. Performing a ratio test on S gives the recurrence-transience boundary:

$$\lim_{n\to\infty} \frac{q_n}{p_n} = \ell\ ,\ \ell > 1 \implies \text{recurrence}\ ,\ \ell < 1 \implies \text{transience.} \tag{1.2}$$

Such a condition can be understood intuitively: if starting the walk at any site m, the ratio of jumps to left is more probable, i.e. $q_{m+n}/p_{m+n} > 1$ as $n \uparrow \infty$, then the random walk is less likely to go further and is being pushed to return to m. Similar qualitative analysis can be done for the transience case.

We are interested in studying the situation where ℓ in eq. (1.2) is approaching 1. For this reason, it is convenient to use the following notation:

$$a_n := \prod_{k=1}^{n} \frac{q_n}{p_n} \text{ where } \frac{q_n}{p_n} := 1 \pm h_n\ ,\ h_n \downarrow 0\ ,\ 0 < q_n, p_n < 1. \tag{1.3}$$

Note that we limit our scope to non-alternating signs for h_n, i.e. $\text{sign}(h_n) = \text{sign}(h_{n-1})$. With this, we want to answer the following questions:

(1) Does $q_n/p_n \downarrow 1$ imply recurrence?

(2) Is it possible if $q_n/p_n \downarrow 1$, the walk would be transient?

(3) Does $q_n/p_n \uparrow 1$ imply transience?

(4) Is it possible if $q_n/p_n \uparrow 1$, the walk would be recurrent?

2. Main Results

We have the following table where (N/A) = Not available, (R) = Recurrent, (T) = Transient, (R/T) = The recurrence depends on a constant, (B) = The walk is pushed backward, and (F) = The walk is pushed forward:

$h_n > 0$	General		$O(1/\log(n))$	$O(n^{-\alpha})$			$O(n^{-1}(\log n)^{\alpha})$
	$\sum h_n = \infty$	$\sum h_n < \infty$		$\alpha < 1$	$\alpha > 1$	$\alpha = 1$	$\alpha > 0$
(B) $1 + h_n$	R	R	R	R	R	R	N/A
(F) $1 - h_n$	N/A	R	T	T	R	T/R	T

Table 1. Summary of Results.

3. Analysis

Proposition 3.1. *With the same notation as in eq.* (1.3), *the following holds:*

(1) $\sum h_n < \infty \implies$ *recurrence.*

(2) $\sum h_n = \infty$ *and* $q_n/p_n = 1 + h_n \implies$ *recurrence.*

Proof. By lemma A.1, if $\sum h_n$ converges, then so is $a_n = \prod(1 \pm h_n)$. If the walk is pushed backward, i.e. the case $1 + h_n$, then a_n converges to a positive value. This implies $S = \infty$, i.e. the walk is recurrent. Similarly, in the case $1 - h_n$, a_n converges to a non-zero value by lemma A.2. The same conclusion holds. In the second case, if $\sum h_n = \infty$, then $a_n \geq 1 + \sum h_n$ diverges to $+\infty$. □

From proposition 3.1, the remaining case to analyze is

$$\textbf{Case (C).} \sum h_n = \infty \text{ and } q_n/p_n = 1 - h_n. \tag{3.1}$$

The recurrence for such a case can not be determined since $\prod(1 - h_n) \downarrow 0$ does not imply that it is summable as shown in proposition 3.3 and table 1. In fact, by [3, theorem 5.1.1], any divergent series $\sum a_n = \infty$ of positive terms can be made smaller without changing its divergent behavior, i.e. $\sum a_n / (\sum_{k=1}^{n} a_k) = \infty$. This suggests that **case (C)** must be studied in a case-by-case basis and may not have a general theory. We study several cases below.

Proposition 3.2. *Let* $h_n = 1/\log(n + 1)$.

(1) $q_n/p_n = 1 + h_n \implies$ *recurrence.*

(2) $q_n/p_n = 1 - h_n \implies$ *transience.*

Proof. The first case follows from proposition 3.1. For the second case, we have

$$\begin{aligned}
\prod_{k=1}^{n}(1 - h_k) &\leq \left(1 - \frac{1}{\log(n+1)}\right)^n = \exp\left(n \log\left(1 - \frac{1}{\log(n+1)}\right)\right) \\
&\leq \exp\left(\frac{-n}{\log(n+1)}\right) \\
&\leq \exp\left(-\sqrt{n}\right).
\end{aligned}$$

Since the right hand side is monotonically decreasing, we may perform the integral test to determine the convergence of $\sum \exp(-\sqrt{n})$. Observe that

$$\int_1^{\infty} \exp(-cy^{\alpha})\, dy = \frac{1}{\alpha\, c^{1/\alpha}} \int_1^{\infty} e^{-u} u^{(1/\alpha)-1}\, du \leq \frac{\Gamma(1/\alpha)}{\alpha\, c^{1/\alpha}} \quad \forall\ c > 0. \tag{3.2}$$

Substituting $\alpha = 1/2$ and $c = 1$ yields the desired result. □

Proposition 3.3. [2, Exercise 5.3.4] *Let* $c > 0$, $p_n = 1/2 + c\,n^{-\alpha}$, $q_n = 1 - p_n$.

(1) $\alpha > 1 \implies$ *recurrent.*

(2) $\alpha < 1 \implies$ *transient.*

(3) $\alpha = 1,\ c \le 1/4 \implies$ *recurrent.*

(4) $\alpha = 1,\ c > 1/4 \implies$ *transient.*

Proof. Observe that we have

$$\frac{q_n}{p_n} = \frac{1/2 - cn^{-\alpha}}{1/2 + cn^{-\alpha}} = 1 - \frac{4c}{n^\alpha + 2c}. \tag{3.3}$$

Case 1. For $\alpha > 1$, the proof follows from lemma A.1 and proposition 3.1.

Case 2. For $\alpha < 1$, the product a_n diverges to 0. Observe that

$$\begin{aligned}\prod_{k=1}^{n}\left(1 - \frac{4c}{k^\alpha + 2c}\right) \le \left(1 - \frac{4c}{n^\alpha + 2c}\right)^n &= \exp\left(n\log\left(1 - \frac{4c}{n^\alpha + 2c}\right)\right)\\ &\le \exp\left(-\frac{4c\,n^{1-\alpha}}{1 + 2c\,n^{-\alpha}}\right)\\ &\le \exp\left(-\frac{4c}{1+2c}\,n^{1-\alpha}\right)\end{aligned} \tag{3.4}$$

The claim holds by similar argument as in eq. (3.2).

Case 3. For $\alpha = 1$, observe that in eq. (3.3), q_n/p_n increases to 1 with rate $1/n$. In particular,

$$0 < c \le \frac{1}{4} \implies \frac{q_n}{p_n} \ge 1 - \frac{1}{n + 2c} \ge 1 - \frac{1}{n} = \frac{n-1}{n}\ \forall\, n > 1 \implies \prod_{k=1}^{n}\frac{q_k}{p_k} \ge \frac{q_1/p_1}{n}. \tag{3.5}$$

The right hand side is not summable. Therefore, the series $\sum\prod q_k/p_k$ diverges, which implies recurrence. On the other hand, the product $\prod q_n/p_n$ can be bounded above as follows

$$\begin{aligned}\prod_{k=1}^{n}\frac{q_k}{p_k} &= \exp\left(\sum_{k=1}^{n}\log\left(1 - \frac{4c}{k+2c}\right)\right)\\ &\le \exp\left(-4c\sum_{k=1}^{n}\frac{1}{k+2c}\right)\\ &\le \exp\left(-\int_1^n \frac{dx}{x+2c}\right)^{4c}\\ &= \log(1+2c)^{4c}\left(\frac{1}{n+2c}\right)^{4c}\end{aligned} \tag{3.6}$$

For $c > 1/4$, the right hand side is summable. $\square$

There are a couple of things that are interesting. First, there is a bifurcation for $\alpha = 1$ at $c = 1/4$. While the effects of c vanishes at ∞, it determines the asymptotic behavior of the walk (whether at ∞ the walk can come back). Since the effects of c is most prominent at small n, this shows how the jump probabilities at sites close to 0 still affects the overall walk, which makes sense since the chain is irreducible (can access site 0 from far away).

Second, the case $\alpha = 1,\ c \le 1/4$ in proposition 3.3 is interesting since the walk is being pushed forward at all sites, but the walk still returns in finite time almost surely. If we push the walk slightly further, will it still be recurrent? In the below case, the answer is negative.

Proposition 3.4. *Let $c > 0$, $p_n = 1/2 + c\,n^{-1}(\log n)^\alpha$, $q_n = 1 - p_n$. If $\alpha > 0$, then the walk is transient.*

Proof. Observe that we have

$$\frac{q_n}{p_n} = \frac{1/2 - c\,(\log n)^\alpha/n}{1/2 + c\,(\log n)^\alpha/n} = 1 - \frac{4c}{n/(\log n)^\alpha + 2c}. \tag{3.7}$$

We apply the same argument as in eq. (3.6):

$$\begin{aligned}
\prod_{k=1}^{n} \frac{q_k}{p_k} &= \exp\left(\sum_{k=1}^{n} \log\left(1 - \frac{4c}{k/(\log k)^\alpha + 2c}\right)\right) \\
&\leq \exp\left(-4c\sum_{k=1}^{n} \frac{1}{k/(\log k)^\alpha + 2c}\right) \\
&\leq \exp\left(-\int_1^n \frac{dx}{x/(\log x)^\alpha + 2c}\right)^{4c} \\
&= \exp\left(-\int_0^{\log n} \frac{e^u\,du}{u^{-\alpha}e^u + 2c}\right)^{4c} = \exp\left(-\int_0^{\log n} \frac{u^\alpha\,du}{1 + 2\,c\,u^\alpha e^{-u}}\right)^{4c} \\
&\lesssim \left(\frac{1}{n}\right)^{4c(1+\alpha)^{-1}(\log n)^\alpha}.
\end{aligned} \tag{3.8}$$

The claim holds. ☐

Note that the right hand side in eq. (3.7) is $q_n/p_n \approx 1 - n^{-1}(\log n)^\alpha$, which means that the walk is pushed further forward compared to the case in proposition 3.3. Also, if we set $\alpha = 0$ in proposition 3.4 and examine the right hand side of eq. (3.8), we recover the result in proposition 3.3 for the transient case.

Appendix A. Some Facts on Infinite Products and Series

Lemma A.1. *Let $\{h_n\}$ be a sequence of non-negative real numbers. The series $\sum h_n$ and the product $\prod(1 \pm h_n)$ either both diverge or both converge.*

Proof. See [4, theorem 2.2.1 and corollary 2.2.3]. ☐

Lemma A.2. *A convergent infinite product has the value 0 if and only if one of the factors is 0.*

Proof. See [3, theorem 3.7.1]. ☐

References

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