Short Excursion in Mathematics through Randomness

MathCats: Mathematics Undergraduate Student Club at The University of Arizona

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University of Arizona

March 25th, 2026

PART I:

“Randomness” in Mathematics

(why the quotation marks?)

Quiz

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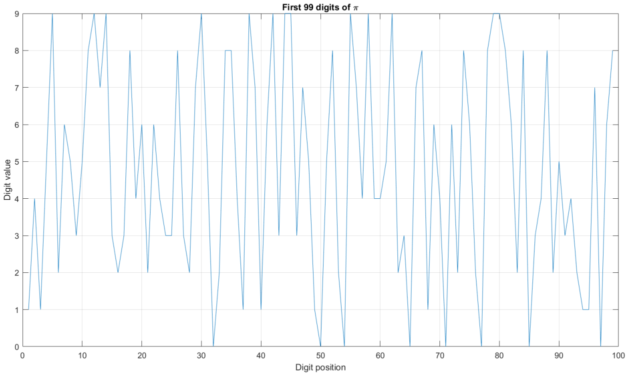

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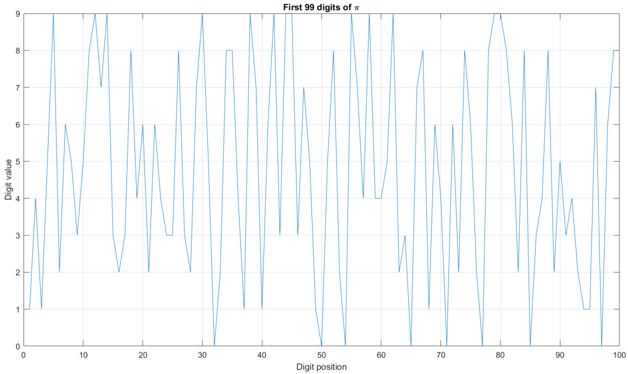

- Digits of π looks random!

Quiz (cont'd)

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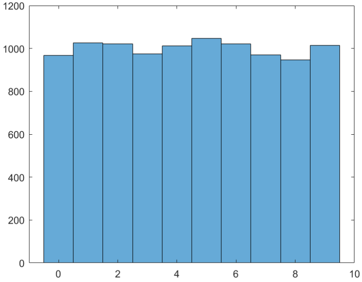
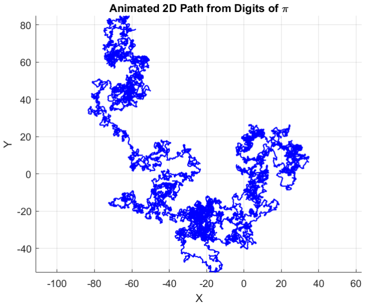


(a) Histogram of 10,000 digits of π.



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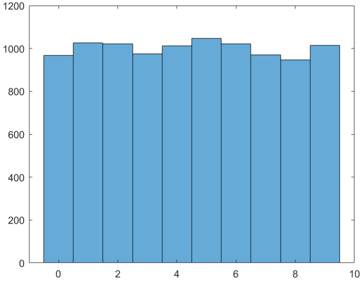

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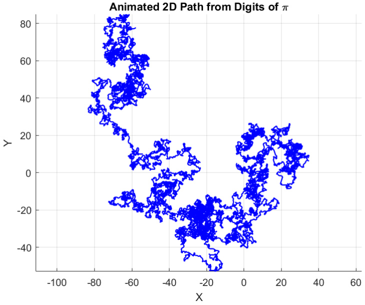

(b) The trajectory of a person's walk if the person were guided by the digits of π.

- Digits of π is roughly uniformly distributed, which is totally random! (see ***entropy*** [CT05])

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- **Open problem:** Show that π is a ***normal number***, i.e. its digits are uniformly distributed.
- Is the distribution of the digits invariant under base-change? e.g. if we transform π to base-2, would the digits still be uniformly distributed?
- What is the distribution of ***normal numbers*** in $[0, 1]$?

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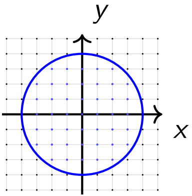
- What is π? If a square were to be morphed into a circle, then it is the scalar (dilation) factor for the area.



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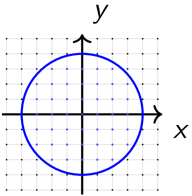

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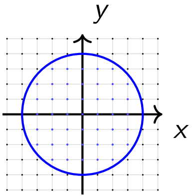

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$$(r^2 - (r - \sqrt{2})^2) \lesssim f(r) - \text{Area}(\mathcal{C}_r) \lesssim c_2\,((r + \sqrt{2})^2 - r^2)$$

$$\frac{2}{r} - \frac{\sqrt{2}}{r^2} \lesssim \frac{f(r) - \text{Area}(\mathcal{C}_r)}{r^2} \lesssim \frac{2 + \sqrt{2}}{r}$$

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$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$
- Properties of the limiting object can be totally different from its underlying approximation. In particular, irrationality (randomness in digits) appears.

Is there a probabilistic way of computing π?

- Sample points $x_1, \ldots, x_N$ independently over a square of length $2r$ in $\mathbb{R}^2$ using uniform distribution.
- Compute the ratio:

$$\frac{\text{number of points landing in circle}}{N} \approx \mathbb{P}(\text{points in } \mathcal{C}_r) = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}.$$

Go to this **link** for simulation.

PART II:

Randomness in Nature

Quiz: Coin Tosses

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In 1828, Robert Brown (a botanist) observed an irregular movement by suspended particles moving in liquid [Edi].

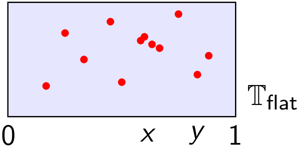

Figure: Suspended particles with initial concentration ρ_0 in a liquid.

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- In 1905, Einstein produced a heuristic computation for the dynamics of suspended particle density [Ein56]. This irregular movement is called ***Brownian Motion***.
- Brownian motion is a building block of algorithms or models in many applications including generative AI (**image generation through diffusion**), **mathematical finance**, and physics.

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- dn is the number of particles passing through a cross-section of width $d\Delta$ over time period τ.
- $\phi(\Delta)$ is the probability of a particle moving by a distance Δ.
- If movement is random, then particles are equally likely to move left or right.

$$dn = n \cdot \phi(\Delta) \cdot d\Delta \qquad , \quad \phi(\Delta) = \phi(-\Delta)$$

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- Expand in Taylor series $\nu(x, t + \tau)$ and $\nu(x + \Delta, t)$.

$$\nu(x, t) + \tau \frac{\partial \nu}{\partial t} + o(\tau) = \nu(x, t) + \mathbb{E}_\phi[\Delta] \cdot \frac{\partial \nu}{\partial x} + \frac{1}{2} \cdot \mathbb{E}_\phi[\Delta^2] \cdot \frac{\partial^2 \nu}{\partial x^2} + o(\Delta^2)$$

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$$\frac{\partial \nu}{\partial t} = \frac{1}{2} \cdot D \cdot \frac{\partial^2 \nu}{\partial x^2} \qquad , \quad D := \frac{\mathbb{V}_\phi[\Delta]}{\tau} \quad \textbf{\textit{(heat equation)}}.$$

$D > 0$ is called the ***diffusion coefficient***.

A Rigorous Derivation of Brownian Motion

Theorem (Donsker's Invariance Principle)

([Don51; KS91]). Let $\{\xi_j \in \mathbb{Z} : j \in \mathbb{N}\}$ *be i.i.d variables with mean zero and finite variance* $\sigma^2 < \infty$ *and* $S_n := \sum_{j=1}^{n} \xi_j$ *with* $S_0 = 0$*. Define the linearly interpolated sequences of random walk* $X_t^{(n)}$ *on* $\mathbb{Z}$

$$X_t^{(n)} := \frac{1}{\sigma\sqrt{n}} Y_{nt} \quad \text{where} \quad Y_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1}\ ,\ t \geq 0.$$

The sequence of paths (functions) $X_t^{(n)}$ *converges in distribution to a standard Brownian motion on the space of continuous functions* $\mathcal{C}([0,\infty))$ *as* $n \to \infty$*.*

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- Any linearly interpolated simple random walk (independent of how the walk is distributed) converges to a BM.
- Path of the digits of π is like a path of BM!

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Thank you and Good luck.

[CT05] Thomas M. Cover and Joy A. Thomas. "Entropy, Relative Entropy, and Mutual Information". In: *Elements of Information Theory*. John Wiley and Sons, Ltd, 2005. Chap. 2, pp. 13–55. ISBN: 9780471748823. DOI: https://doi.org/10.1002/047174882X.ch2. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/047174882X.ch2. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/047174882X.ch2.

[Don51] Monroe D. Donsker. "An invariance principle for certain probability limit theorems". In: *Memoirs of the American Mathematical Society* 6 (1951), pp. 1–12. URL: https://search.worldcat.org/title/An-invariance-principle-for-certain-probability-limit-theorems/oclc/184761671.

[Edi] Britannica Editors. *Robert Brown*. URL: https://www.britannica.com/biography/Robert-Brown-Scottish-botanist.

[Ein56] A. Einstein. *Investigations on the Theory of the Brownian Movement*. Dover Books on Physics Series. Dover Publications, 1956. ISBN: 9780486603049. URL: https://books.google.com/books?id=AOIVupH_hboC.

[HC52] David Hilbert and Stephan Cohn-Vossen. *Geometry and the Imagination*. Translated by P. Nemenyi. New York: Chelsea Publishing Company, 1952.

[KS91] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics (113) (Book 113). Springer New York, 1991. ISBN: 9780387976556. URL: https://books.google.com/books?id=ATNy_Zg3PSsC.